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KOLMOGOROV STABILITY, THE IMPOSSIBILITY OF FERMI ACCELERATION AND THE EXISTENCE OF PERIODIC SOLUTIONS IN SOME HAMILTONIAN-TYPE SYSTEMS†

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In certain special systems, pertaining to various branches of mechanics and physics, there is no explicitly occurring small parameter, but a suitable change of variables will artificially introduce a small parameter into the equations of motion. One can then use the Kolmogorov–Arnol'd–Moser Theorem to prove the existence of invariant (Kolmogorov) tori, which, moreover, fill the whole of phase space except for a set of finite measure (the measure of the whole phase space is infinite). Bounds are obtained for the deformation of invariant tori and the relative measure of the Kolmogorov set. It follows from Poincaré's Geometric Theorem that in all problems under consideration there exist infinitely many periodic solutions. The results of numerical experiments in connection with Ulam's problem receive their first comprehensive, rigorous justification, and analogous propositions are derived for other situations.

The case of pendulum-type systems is then considered in this context by treating the reciprocal of the momentum as the small parameter. This device was first used in connection with the equations of mechanics (when momentum = angular velocity) to determine the asymptotic behaviour of fast rotations [1]. In the case of Ulam's problem a small parameter reciprocal to the particle velocity was introduced in [2].

This paper generalizes and develops results of [3] which follows from Theorem 1 as a special case.

INTRODUCTION

THE RANGE of problems considered here is intimately connected with the question of whether an acceleration mechanism analogous to Fermi's stochastic mechanism [4] can arise in deterministic systems, and with an analysis of the stability of conservative systems [5].

Complicated questions already arise with respect to the equations

$$\varphi^{\cdot\cdot}+\omega_0^2 \sin\varphi=F(t) \tag{0.1}$$

$$x^{\cdot\cdot}+\omega_0^2(1+\alpha x^2)x=F(t), \qquad \alpha>0 \tag{0.2}$$

which describe oscillators driven by forces $F(t)$ containing many harmonics. The semi-qualitative theory and numerical experiments [5, 6] imply that, if F satisfies certain conditions, the oscillator will vibrate as if driven by a random force; in other words, the mechanism in operation will be analogous to Fermi acceleration and the energy of the vibrations will increase on average in direct proportion to time.

In Ulam's model [7] of Fermi acceleration, a particle moves between two parallel walls which are oscillating periodically; the particle rebounds elastically from each wall. Numerical experiments, however, have shown that no acceleration takes place, and the phase plane of the appropriate mapping is divided into three regions: (1) a low-velocity region, where the motion is stochastic; (2) an intermediate-velocity region, containing islands of stability around elliptic points within the stochastic components; (3) a high-velocity region, with narrow stochastic layers in the neighbour-

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hood of the separatrices are isolated from one another by invariant curves which span the entire phase range (see [8]). A partial explanation of this phenomenon was given by Zaslavskii and Chirikov in 1964. At the same time, the possibility remained that in certain situations the particle may accelerate to high velocities [9]. Several publications have investigated a simplified Ulam model: the oscillating walls are replaced by fixed walls which, however, react on the bouncing particle as if they were oscillating periodically. The results obtained in studies of Ulam's models, exact and simplified, are analogous.

It will be shown below, in particular, in regard to equations (0.1) and (0.2) with a smooth periodic perturbation F and in regard to Ulam's models, that acceleration to high velocities is impossible, and that the Fermi mechanism may operate only over a limited region of the phase space. An explanation will also be given for the experimentally observed structure of the phase plane in Ulam's problem (Theorem 2). As a corollary of Poincaré's Geometric Theorem it will follow that there exist infinitely many periodic solutions; this will be proved without assuming that the equations satisfy the traditional symmetry conditions [10–13]. Applied, in particular, to one of the most popular subjects of study—Beletskii's equation—the results are the strongest possible (see the survey article [14]). Moreover, under suitable symmetry conditions, for any relatively prime numbers $m>0$, n the methods of [10, 12, 13] enable one to prove the existence of one $2\pi m/n$-periodic solution, while by Theorem 3 there are at least two such solutions.

The impossibility of acceleration is due to the existence of an adiabatic invariant and follows from the existence of invariant curves of the recurrence mapping. In regard to Ulam's problem, this was previously pointed out in some physical studies, but rigorous results were lacking up to [15]. It was concluded [16] that acceleration cannot occur in the exact Ulam model, but the question of the measure of the invariant curves was not discussed and the proof was needlessly complicated.

Ideas very similar to those put forward here may be found in [17–20], where related results were obtained (without estimating the measure of the Kolmogorov set). Among the topics studied are the equation $x^{\cdot\cdot}+2x^3=p(t)$ with a periodic piecewise continuous function $p(t)$ [17]; equations of Duffing's type [18] (see Sec. 2); a special case of pendulum-type systems [19, 20] which has bearings on problems of mechanics (see Sec. 2). Authors have hitherto concentrated on determining the minimum smoothness necessary for the application of KAM-theory [18, 20], or on proving the existence of acceleration in a pendulum-type system in the "non-conservative" case [19] (see Sec. 2). By studying equations which are periodic in the time t, of the type $x^{\cdot\cdot}+f(t,x)=0$ under some very general assumptions and applying the Poincaré–Birkhoff Theorem [21] to a certain recurrence mapping, Jacobowitz [22] proved an analogue of parts 2 and 3 of our Theorem 3. With regard to the equation $x^{\cdot\cdot}+g(x)=p(t)$, where $g(x)\operatorname{sign}(x)$ increases more rapidly than $|x|$ as $x\to\infty$, Ding [23] was also able to prove, subject to a symmetry assumption, that all the solutions are bounded; the proof relied on topological arguments concerning the recurrence mapping and did not use KAM-theory.

Our main attention in this paper will be devoted to the measure of the Kolmogorov set and the extent of deformation of invariant tori, on the assumption that the system is sufficiently smooth. In Secs 1 and 2 we will describe a unified approach to systems of three types, which were considered separately in [16–20]; incidentally, the proofs proposed in those papers may be simplified considerably by using known results on the accuracy of the averaging procedure.

There are rigorous proofs [24–28] that the acceleration phenomenon is possible and indeed typical in various other problems. However, the phases for the trajectories constructed (or quantities analogous to the phases) are strongly correlated, offering some contrast to the stochastic mechanism and indicating that no mixing occurs. A similar result will be obtained below in Sec. 6 for a mixed Ulam model.

1. BRIEF DESCRIPTION OF THE MAIN METHODS

Once a small parameter has been artificially introduced, the systems considered here assume the form of Hamiltonian systems with one rapidly rotating phase and two slow variables [2], the role of the fast phase being played by the coordinate or the time. The first situation is that of the systems studied in Sec. 2; systems of the second type arise in Sec. 3. Similarly, the Ulam mappings $z \to z + \varepsilon l(z) + O(\varepsilon^2)$, $z = (p, q)$ in Sec. 4 are shown to be exact canonical mappings, close to the identity mapping; they may therefore be represented [2] as recurrence mappings for 2π-periodic Hamiltonian systems of the second type with a Hamiltonian $O(\varepsilon^2)$-close to εE, where

$$E(z) = \frac{1}{2\pi} \int^{z} l^{(q)} dp - l^{(p)} dq$$

and $l^{(p)}$, $l^{(q)}$ are the components of the shift l, the integral appearing in the definition of E is independent of the integration path (a non-canonical version of this statement will be used in Sec. 6).

In the analytic case the canonical averaging procedure makes it possible to transfer the dependence of the Hamiltonian from the fast variable to the exponentially small terms (for details see [2, 29]), that is, to find a close integrable system. Then, considering the corresponding recurrence mapping over a period, one uses Moser's theorem on invarient twist curves or (in greater generality) small twist curves [30]. Bounds have been established [31] for the measure and deformation in the case of twists, but the results carry over without change to the general case of small twists just as Moser's original proof [30] (which provided the basis for [31]) also carries over. The trajectories that pass through an invariant curve of the recurrence mapping form the required torus.

Henceforth $C_1, \ldots, C_{18}$ will be constants, $\langle \cdot \rangle^{\varphi}$ the operation of averaging over the angular variable φ and $\{\cdot\}^{\varphi} = (\cdot) - \langle \cdot \rangle^{\varphi}$ the operation isolating the purely periodic part. The symbol $\|\cdot\|_{C^s}$ denotes the C^s-norm, where s is a natural number of ω and C^{ω} is the space of functions analytic in some complex neighbourhood of a real domain.

2. PENDULUM-TYPE SYSTEMS AND DUFFING-TYPE EQUATIONS

Pendulum-type system. On the direct product

$$M = \mathbf{R}^1\{y\} \times S^1\{x \bmod 2\pi\} \times S^1\{t \bmod 2\pi\}$$

we consider the system $x^{\bullet} = \partial H/\partial y$, $y^{\bullet} = -\partial H/\partial x$ with Hamiltonian

$$H = \sum_{i \geqslant 0} c_i(t, x)\, y^i \tag{2.1}$$

where c_i are analytic functions on a torus $T^2\{t, x\}$. Let $c_i = 0$ for $i > n$ (in mechanics $n = 2$) and assume that c_n is independent of x, $c_n(t) > \delta > 0$ for all t.

Duffing-type equations. On the direct product

$$M = \mathbf{R}^2\{x, y\} \times S^1\{t \bmod 2\pi\}$$

we consider the system with Hamiltonian

$$H = y^2/2 + ax^n + f(x, t)$$

where $f(x, t)$ is a polynomial in x of degree at most $n - 1$ with coefficients which are 2π-periodic and analytic in t, $a > 0$, $n \geqslant 4$ is even.

In particular, for Duffing's equation $n = 4$, $f(x, t) = bx^2 + p(t)x$.

We now transform from canonically conjugate variables x, y to action-angle variables I, φ for the Hamiltonian $H_0 = y^2/2 + ax^n$; I, φ may be chosen so that

$$x = I^{2/(n+2)} X(\varphi), \quad y = I^{n/(n+2)} Y(\varphi)$$

where X, Y are analytic functions on

$$S^1\{\varphi \bmod 2\pi\}, \quad I = C_1 H^{(n+2)/(2n)}$$

$$C_1 = \frac{2\sqrt{2}}{\pi} \int_0^{a^{-1/n}} \sqrt{1 - ax^n}\, dx > 0$$

In the new variables the Hamiltonian is

$$H = C_1^{-n\alpha} I^{n\alpha} f(I^\alpha X(\varphi), t), \quad \alpha = 2/(n+2) \tag{2.2}$$

Thus, the Hamiltonians (2.1) and (2.2) are special cases of the Hamiltonian

$$H = c_n(t)\, I^{r\alpha} + \sum_{i=0}^{r-1} c_i(t, \varphi)\, I^{i\alpha} \tag{2.3}$$

where $\alpha = 1$ for pendulum-type systems and $\alpha = 2/(n+2)$ for Duffing-type equations.

We introduce new variables $J = \varepsilon I$, $\tau = \varepsilon^{-\beta} t$, $\beta = n\alpha - 1$, using a small parameter $\varepsilon > 0$. The system

$$\varphi^{\cdot} = \partial H / \partial I, \quad I^{\cdot} = -\partial H / \partial \varphi$$

becomes

$$d\varphi / d\tau = \partial F / \partial J, \quad dJ / d\tau = -\partial F / \partial \varphi$$

which has the Hamiltonian

$$F = c_n(t)\, J^{n\alpha} + \sum_{i=0}^{n-1} c_i(t, \varphi)\, J^{i\alpha} \varepsilon^{(n-i)\alpha}$$

and contains the slowly varying parameter t (since $\beta > 0$).

According to [29], there exists a canonical substitution $(J, \varphi) \to (J', \varphi')$, which is $O(\varepsilon^\alpha)$-close to the identity and dependent on the time t, which reduces the Hamiltonian to the form $F_0(J', t) + F_1(J', \varphi', t)$, where $F_I = O(\exp(-C_2 \varepsilon^{-\beta}))C_2 > 0$. Let T_0 (respectively, T) be the recurrence mapping of the system with Hamiltonian F_0 $(F_0 + F_1)$ over the period $2\pi\varepsilon^{-\beta}$. Then, in terms of z, φ' coordinates, where $J' = C_3 + \varepsilon^\beta z$, $C_3 \neq 0$, the mapping T is exponentially close to T_0: $z_1 = z$, $\varphi_1' = \varphi' + g(z)$, and

$$\frac{\partial g}{\partial z} = \int_0^{2\pi} \frac{\partial^2 F_0(J, \lambda)}{\partial J^2}\, d\lambda$$

Applying Moser's theorem and returning to I, φ coordinates, we obtain the following result.

Theorem 1. Let $n\alpha > 1$, $c_n(t) > \delta > 0$, $C_4 > 0$. Then the whole toroidal layer $|I - I_0| < C_4$, except possibly a set of relative measure $O(\exp(-C_5 |I_0|^\beta))$ $(C_5 > 0)$, will be covered by Kolmogorov tori. Each of the Kolmogorov tori will be enclosed between the tori

$$I = I_{1,2}, \quad |I_2 - I_1| = O((|I_0| + 1)^{1-\alpha}). \tag{2.4}$$

Corollaries. 1. The entire phase space M, except possibly a set of finite measure, is filled by Kolmogorov tori, which carry conditionally periodic motions.

2. During the motion the variable I may experience only fluctuations of size $O(|I|^{1-\alpha} + 1)$ in accordance with (2.4).

Remarks. 1. If the functions c_i appearing in the expression for the Hamiltonian (2.3) have only a finite but sufficiently large number of derivatives with respect to t, φ, Theorem 1 and its corollaries remain valid, but the exponentially small estimate must be replaced by a power estimate: $O(|I_0|^{-m})$.

2. Let us assume that the coefficient $c_n(t, \varphi)$ of the leading term in (2.3) depends on φ but is μ-close in the $C^s(s \leq \omega)$-norm to a positive function that is independent of φ. Then, if s is sufficiently large, Theorem 1 remains valid (but see Remark 1), with the estimate (2.4) replaced by the weaker estimate

$$|I_2-I_1|=O(\mu|I_1|+|I_0|^{1-\alpha}+1)$$

This is the situation for Duffing-type equations if H_0 (the leading terms of H) is $H_0=k(t)/2y^2+a(t)x^n$, where $k(t)$ and $a(t)$ are close in the C^s-norm to positive constants.

3. If the functions $c_n, \ldots, c_{n-k+1}$ are independent of φ, the estimate (2.4) may be sharpened: $|I_2-I_1|=O[(|I_0|+1)^{1-k\alpha}]$.

In particular, for the Duffing equation itself $n=4$, $f(x,t)=bx^2+p(t)x$ and the variable $I=C_1H_0^{3/4}$ may experience oscillations only of order $O(|I|^{1/3}+1)$.

We will now describe some examples of pendulum-type systems encountered in mechanics.

1. *Oscillations of a mathematical pendulum with vibrating point of suspension* [32]:

$$H=y^2/2-\omega_0^2(1+f(t))\cos x \tag{2.5}$$

where $f(t)$ is a 2π-periodic function, x is the angle of deflection and y is the angular velocity of rotation of the pendulum. An equation of the type $x^{\bullet\bullet}+\omega_0^2(1+f(t))\sin x=0$ is also encountered in other problems. The system is integrable if either $\mu_1=\omega_0^2=0$ or $\mu_2=\|\{f(t)\}\|C^\omega=0$.

2. *Two-dimensional oscillations of a satellite in elliptic orbit (Beletskii's equation)* [33]:

$$H=\frac{1}{2}\left[\frac{p}{1+e\cos\nu}-2(1+e\cos\nu)\right]^2-(1+e\cos\nu)\mu\cos\delta \tag{2.6}$$

Here the roles of x, y, t are taken, respectively, by variables δ, p and ν that have the following meaning: ν is the true anomaly of the centre of mass of the satellite in an elliptic orbit of eccentricity e, $\delta/2$ is an angle characterizing the rotation of the satellite about the radius vector drawn from the attractive centre to the centre of mass of the satellite and $p=(2+d\delta/d\nu)(1+e\cos\nu)^2$ is a constant times the angular velocity of rotation of the satellite about its centre of mass. The quantity μ characterizes the mass distribution of the satellite, $|\mu|<3$. The problem is integrable if $\mu_1=\mu=0$ or $\mu_2=e=0$.

3. *The restricted problem of the revolution of a symmetric heavy solid with a fixed point* [34]. The Euler–Poisson variables in the limit may be expressed in terms of the quantities ξ mod 2π and η corresponding to the x, y variables.

$$H=\eta^2/2+(2h)^{-1}(\alpha\cos\xi+\beta\sin\xi\sin\tau) \tag{2.7}$$

Here the role of time is played by $\tau=\sqrt{(2ht')}$; $\alpha^2+\beta^2=1$. The numbers α, β and h characterize the energy and area integrals of the system and η is the projection r of the angular velocity of the body onto the axis of dynamic symmetry, divided by $\sqrt{2h}$. The system is integrable if $\mu_1=(2h)^{-1}=0$ or $\mu_2=\beta=0$.

4. *Motion of a mathematical pendulum driven by a periodic torque with zero mean.*† The equation is $q^{\bullet\bullet}+\omega_0^2\sin q=f(t)$, where q is the angle made by the pendulum with the vertical, $f(t)$ is a 2π-periodic torque and $\langle f\rangle^t=0$. This equation may be expressed in Hamiltonian form with a Hamiltonian

$$H={}^1/_2(p+f_1(t))^2-\omega_0^2\cos q, \tag{2.8}$$

where $q^\bullet=p+f_1(t)$, $f_1(t)=\int f(t)dt+C_4$, the constant C_4 is such that $\langle f\rangle^t=0$. The problem is integrable if $\mu_1=\omega_0^2=0$ or $\mu_2=\|f_1(t)\|_C=0$.

5. *The motion of a mathematical pendulum suspended on a cord of periodically varying length (Einstein's pendulum):*

$$H=p^2/(2l^2)-\omega_0^2\cos q \tag{2.9}$$

†BUROV A. A., Some problems in the dynamics of pendulum-type systems. Candidate dissertation, Moscow State University, Moscow, 1984.

where q is the angle of deflection, $p = l^2 q^{\cdot}$ is the conjugate momentum and $l(t)$ is the length of the cord. The system is integrable if $\mu_1 = \omega_0^2 = 0$ or

$$\mu_2 = \| \{l(t)\} \|_{C^\omega} = 0.$$

Note that examples 1, 4 and 5 may be combined.

In all the examples the relative measure of the complement M^* of the Kolmogorov set is at most $O(\exp(-C_5|y_0|)$. In examples 1–3 and 5 the y variable is the canonical conjugate of the angle of deflection, i.e., the angular momentum, which is conserved to within $O(|y_0|^{-1})$ (see Remark 3).

Example 4 has an interesting generalization. Add a term $-xp(t)$ which is 2π-periodic in t and non-periodic in x to the Hamiltonian (2.1); this term characterizes a driving force $p(t)$. In the "conservative" case, when $\langle p \rangle^t = 0$, when the equations of motion can be written in Hamiltonian form with a Hamiltonian (2.1), proving the existence of a Kolmogorov set and the absence of acceleration. In the "non-conservative" case ($c = \langle p \rangle^t \neq 0$) it can be proved that there are no invariant tori near the levels $y \equiv$ const, and acceleration occurs in all trajectories in the domain of large positive cy (cf. [19]). A similar result for a mixed Ulam model will be established in Sec. 6.

The Hamiltonians (2.5)–(2.9) share the following property: if $\mu_1 = 0$ the dependence on the phase x disappears; if $\mu_2 = 0$ one obtains the Hamiltonian of a mathematical pendulum

$$H = {}^1\!/_2 y^2 + \omega_0^2 \cos x, \quad \omega_0^2 = \mu_1$$

(to prove this for (2.6), first replace p by $p+2$). Using this, one can verify the following facts.

1. If $C_4 > 0$, the numbers $2\sqrt{|\mu_1|}$, $3\sqrt{|\mu_1|}$ are not integers and μ_2 is small, then the relative measure of the set M^* in the annulus $|y - y_0| < C_4$ is at most $O(\sqrt{\mu_2} \exp(-C_5 z)$ where $z = |y_0| + 1$; the invariant tori will be $O(|\mu_2| z^{-1} + \sqrt{|\mu_1|} \exp(-C_5 z))$-close to the surfaces

$$ {}^1\!/_2 y^2 + \mu_1 \cos x = \text{const} \tag{2.10}$$

In Examples 4 and 5, when the Hamiltonian is $H = {}^1\!/_2 \alpha(t) y^2 + f_1(t) y - \omega_0^2 \cos x$, the last estimate may be sharpened to $O(|\mu_2| z^{-2} + \sqrt{|\mu_2|} \exp(-C_5 z))$. The constant $C_5 > 0$ and the O-estimates depend on μ_1 and C_4.

Thus, the full measure of M^* is $\operatorname{mes} M^* = O(\sqrt{|\mu_2|}$ (to obtain estimates in the domain of small $|y|$ the phase space is divided into domains of three types: neighbourhoods of elliptic periodic solutions, neighbourhoods of separatrices of hyperbolic periodic solutions, and annular domains complementary to domains of the first two types. For domains of the third type the required estimates follow from [31]; for domains of the second—from [35].

The proof for domains of the first type uses the fact that an integrable system is reducible to Birkhoff normal form, one step of the Birkhoff normalization procedure for the perturbed system and estimates for the invariant curves of small twists.)

2. If $\mu_1 \neq 0$ and $2\sqrt{|\mu_1|}$ or $3\sqrt{|\mu_1|}$ is an integer, the above estimates remain valid outside an arbitrary neighbourhood of the elliptic periodic solution $x \equiv 0 \bmod 2\pi$ (for $\mu_1 < 0$) or $x \equiv \pi \bmod 2\pi$ (for $\mu_1 > 0$).

3. For small μ_1 the relative measure of M^* in the domain $|y - y_0| < C_4$ is at most $O(\sqrt{|\mu_1|} \exp(-C_5 z))$, and the invariant tori are $O(|\mu_1| z^{-1} + \sqrt{|\mu_1|} \exp(-C_5 z))$—close to the tori y = const. The constant $C_5 > 0$ and the O-estimates depend on μ_2 and C_4. Thus $\operatorname{mes} M^* = O(\sqrt{|\mu_1|})$.

4. For small μ_1 and μ_2 the relative measure of M^* in the domain $|y - y_0| < C_4$, $|y_0| > 2C_4$ is at most $O(\sqrt{|\mu_1 \mu_2|} \exp(-C_5 |y_0|))$, and the invariant tori are $O(|\mu_1 \mu_2| z^{-2} + \sqrt{|\mu_1 \mu_2|} \exp(-C_5 |y_0|))$-close to the surfaces (2.10). For example 4 and example 5 this estimate may be sharpened to $O(|\mu_1 \mu_2| z^{-2} + \sqrt{|\mu_1 \mu_2|} \exp(-C_5 |y_0|))$. The constant $C_5 > 0$ and the O-estimates depend on C_4.

It will be proved in Sec. 3 that in the domain $|y_0| < C_4$ the measure of M^* is also $O(\sqrt{|\mu_1 \mu_2|})$, and the invariant tori are close to surfaces analogous to (2.10). Thus $\operatorname{mes} M^* = O(\sqrt{|\mu_1 \mu_2|})$.

3. INVESTIGATION OF EXAMPLES 1–5 IN THE DOMAIN OF SMALL μ_1 AND μ_2 AND SMALL MOMENTUM y

The Hamiltonians (2.5)–(2.9) may be written (to within a negligible term)

$$H = {}^1/_2 y^2(1+g(t)) + h(t)y - \omega_0^2(\cos x + f(t,x))$$

where the 2π-periodic functions g, h and f are analytic functions of μ_2, with $g(t)\equiv 0$, $f(t,x)\equiv 0$, $h(t)\equiv$const when $\mu_2 = 0$.

Setting

$$x = x' + \int \{h\}^t dt + y \int \{g\}^t\, dt, \quad y = y' - \langle h\rangle^t/(1+\langle g\rangle^t)$$

we introduce a time-dependent canonical change of variables $(x, y)\to(x', y')$. In the new variables, the system is described by the Hamiltonian

$$H = {}^1/_2 G y'^2 - \omega_0^2(\cos x' + l(t, x', y'))$$

where $l\equiv 0$ if $\mu_2 = 0$; $G = 1 + \langle g\rangle^t$. Introducing a new variable $Y = \varepsilon^{-1}y'$, where $\varepsilon>0$ is a small parameter, we obtain a Hamiltonian system with canonically conjugate variables x', Y and Hamiltonian

$$F = \varepsilon^{-1}H = \varepsilon\{{}^1/_2 G Y^2 - (\omega_0/\varepsilon)^2(\cos x' + l(t, x', \varepsilon Y))\}$$

Let C_6 and C_7 be positive constants and $|\omega_0/\varepsilon|<C_6$. It follows from [2] that for small ε and μ_2 there is a canonical change of variables, $O((\omega_0/\varepsilon)^2\mu_2)$-close to the identity, which is defined in a complex neighbourhood of the real domain $|Y|<C_7$ and reduces F to the form

$$F'(Y, x') + F''(Y, x', t)$$

$$F'' = O((\omega_0/\varepsilon)^2\mu_2 \exp(-C_8\varepsilon^{-1})) \quad (C_8>0)$$

Hence it follows that for small $\mu_1 = \omega_0^2$, μ_2, C_4 the entire toroidal layer $|y|<C_4$ except possibly a set of relative measure $O(\sqrt{|\mu_1\mu_2|}\exp(-C_9\varepsilon^{-1}))$, where $\varepsilon = \max\{|\omega_0|;\ C_4\}$, $C_9>0$, is covered by invariant tori of the system which are perturbations of the surfaces $Gy^2/2 - \omega_0^2\cos x$ = const (compare with [2], Sec. 6).

4. ULAM'S MODELS: EXACT AND SIMPLIFIED

The exact model [2]. Choose the x axis along the trajectory of the particle and walls. Assume that the coordinates of the walls are 2π-periodic smooth functions $d_1(t)$, $d_2(t)$, $d_1<d_2$, where t is the time. In the interval between two collisions the particle moves uniformly in a straight line, governed by the law $x(\tau) = v(\tau - t)$, where v is the velocity and t plays the role of the instant of time at which the particle passes the origin. Suppose that after collision with the second wall v and t take values $(-v', t')$. Thus, for $v>\max_t|d_2^{\bullet}(t)|$, there is a well-defined mapping A_2: $(v, t)\to(v', t')$. A similar definition yields a mapping A_1: $(v', t')\to(v'', t'')$, corresponding to collision with the first wall.

Investigation of the model reduces to studying what is known as the "exact" Ulam mapping $A = A_1\circ A_2$, which is defined for $v>v_{cr} = 2\max_t|d_1^{\bullet}| + 2\max_t|d_2^{\bullet}|$ and maps the half-cylinder $\mathbf{R}_+^1\{v>v_{cr}\}\times S^1\{t \bmod 2\pi\}$ into the half-cylinder $K = \mathbf{R}_+^1\{v: v>0\}\times S^1\{t \bmod 2\pi\}$. It is readily shown that A conserves the relative integral invariant v^2dt, i.e. it is exact canonical in the coordinates $I = v^2$, $t \bmod 2\pi$ (cf. [2, 8]).

We alter the time-scale by putting $t = \varepsilon\tau$, $v = \varepsilon^{-1}V$. The mapping A: $(J, t)\to(J, t)$, where $J = V^2/2$, in the domain $0<C_{10}<V<C_{11}$ is exact canonical and $O(\varepsilon)$-close to the identity [2], and the role of the function E is played by $E = \pi^{-1}L$ [2], where $L = V(d_2 - d_1)$ is a well-known adiabatic invariant [36]. Therefore, if the d_is are analytic functions then, applying the averaging procedure and estimates for the measure and deformation of the invarient small twist curves and returning to the original variables, we obtain the following result.

Theorem 2. Let $C_4>0$. Let mes_i be two natural measures defined on the half-cylinder K by the two-forms $dv^i\wedge dt$, $i = 1, 2$. Then the entire annulus $|v - v_0|<C_4$, except possibly a set of relative measure $\text{mes}_i = O(\exp(-C_{12}v_0))$ $(C_{12}>0)$, is covered by invariant curves of the exact Ulam mapping. Each such curve is enclosed between level curves $L = L_{1,2}$ of the adiabatic invariant $L = v(d_2 - d_1)$ for which

$$|L_2 - L_1| = O(1) \tag{4.1}$$

Corollaries. 1. The entire domain of definition of the exact Ulam mapping, except possibly a set of finite measure mes_i, is covered by invariant curves.

2. During the motion, the adiabatic invariant L of the particle oscillates to the order of $O(1)$.

Ulam's mapping A is obviously integrable if $d_1 \equiv \text{const}$ and $d_2 \equiv \text{const}$. Fix $D = \langle d_2 \rangle' - \langle d_1 \rangle' > 0$ and introduce a small parameter $\mu = \| \{d_1\}' \|_{C^\omega} + \| \{d_2\}' \|_{C^\omega}$. It is easy to see that the mapping $A:(J, t) \to (J, t)$ will be $O(\mu\varepsilon)$-close to the twist mapping $J \to J$, $t \to t + 2\varepsilon D V^{-1}$. Therefore, for $v_0 > C_4 > 0$ the estimates in the theorem become respectively $O(\sqrt{\mu}\exp(-C_{12} v_0))$

$$|L_2 - L_1| = O(\mu + \sqrt{\mu}\exp(-C_{12} v_0)) \tag{4.2}$$

To investigate the domain $v_0 > C_4$ we apply the substitution $v = \varepsilon V$, where $0 < C_{13} < V < C_{14}$, $\varepsilon > 0$ is a small parameter. If μ/ε is large, A is not well defined (the particle may collide twice in succession with the same wall). We shall therefore assume from now on that $\mu < C_{15}\varepsilon$, $C_{15} > 0$. After the second substitution $V = V_0(1 + \varepsilon w)$, $|w| < C_{16}$, the mapping A becomes

$$w' = w + O\left(\frac{\mu}{\varepsilon^2}\right)$$

$$t' = t + \frac{2D}{\varepsilon V_0} - \frac{2D}{V_0} w + O_w(\varepsilon) + O\left(\frac{\mu}{\varepsilon^2}\right)$$

where O_w is a term independent of t. Applying the estimates of [31] and returning to the original coordinates (v, t), we see that for small μ/ε^2 the entire annulus $C_{13}\varepsilon < v < C_{14}\varepsilon$, except possibly a set of relative measure $O(\sqrt{\mu}/\varepsilon)$, is filled by invariant closed curves which are $O(\sqrt{\mu\varepsilon})$-close to the circles $v = \text{const}$ and, moreover, the $O(\sqrt{\mu\varepsilon})$-neighbourhood of each circle $v = \text{const}$ contains such a curve.

Corollaries. 1. If the velocity of the particle at some instant of time is very small, it will never exceed a lower threshold $O(\sqrt{\mu})$.

2. The entire domain $0 < v < C_4$, except possibly a set of measures $\text{mes}_1 = O(\sqrt{\mu}\ln\mu)$, $\text{mes}_2 = O(\sqrt{\mu})$, is covered by invariant closed curves of the exact Ulam mapping. The same estimates hold for the domain $0 < v < +\infty$.

3. If the velocity of the particle at some instant of time is such that $v_0 > C_{17}\sqrt{\mu}$, where $C_{17} > 0$ is a constant, then during the motion the velocity will only oscillate by a relative amount of $O(\sqrt{\mu})$.

The simplified model. Let us assume now that the walls are stationary, at coordinates $d_1^0 < d_2^0$, but when the particle collides with them they act on it as if they were oscillating as described by $x = d_i(t)$, where $d_1(t)$, $d_2(t)$ are smooth functions. Corresponding to this simplified model we have what is known as the simplified Ulam mapping A, which is constructed in the same way as the exact mapping. This mapping A is exact canonical in coordinates $I = v, t$ (cf. [8]). After the substitution $v = \varepsilon^{-1} V$ the mapping $A:(V, t) \to (V, t)$ becomes exact canonical and $O(\varepsilon)$-close to the identity. A simple calculation shows that the function E in this case may be defined as $E = \pi^{-1}(d_2 - d_1 + D\ln V) = \pi^{-1} D \ln\varepsilon + \pi^{-1} G$, where $G = d_2 - d_1 + D\ln v$, $D = d_2^\circ - d_1^\circ$ is the distance between the walls. It is obvious that all the arguments and estimates in regard to the exact Ulam mapping go through without change. In the estimates (4.1) and (4.2) we now have

$$L = \exp(G/D) = v\exp((d_2 - d_1)'/D).$$

5. PERIODIC SOLUTIONS

The existence of the Kolmogorov set and Poincaré's Geometric Theorem [36] imply the following theorem.

Theorem 3. If n, m are relatively prime numbers, $m > 0$, then any pendulum-type system has at least two distinct $2\pi m$-periodic solutions such that $x(2\pi m) = x(0) + 2\pi n$.

2. For Duffing-type equations we demand in addition that $n/m > C_{18} > 0$. In that case there exist two $2\pi m$-periodic solutions such that $\varphi(2\pi m) = \varphi(0) + 2\pi n$.

3. Let A be the (exact or simplified) Ulam mapping. If n and m are relatively prime numbers such that $m/n > C_{18} > 0$, then the mth iteration A^m will have at least $2m$ fixed points such that $I \to I$, $t \to t + 2\pi n$ (for large I the mapping A has only a slight effect on t; that is why the expression $t \to t + 2\pi n$ has an exact meaning). Correspondence to these are two $2\pi n$-periodic motions of the particle.

Remark. The Kolmogorov tori lie in the closure of the set of periodic solutions [18]. Therefore, by Theorems 1 and 2, the entire phase spaces of the systems under consideration, with the possible exception of a set of finite measure, are densely filled with periodic trajectories.

6. THE MIXED ULAM MODEL

One can combine the exact and simplified models, assuming that the walls oscillate according to a law $x = d_i^\circ(t)$, but upon collision act as if they were oscillating according to a law $x = d_i(t)$. The arguments presented below also hold if $d_i^\bullet(t)$ are 2π-periodic functions with non-zero averages. A relativistic mixed Ulam model has been considered previously [28].

Theorem 4. If k is the quotient $v(t+2\pi)/v(t)$ of the velocities of the particle at times 2π apart, then

$$|k| = \exp\left(-\oint \frac{d_2^\bullet - d_1^\bullet}{d_2^\circ - d_1^\circ}\, dt\right) + O\left(\frac{1}{v(t)}\right)$$

(by the velocity at the instant of collision we mean the velocity immediately before or after the collision).

Proof. It can be shown that the corresponding mapping A: $(V, t) \to (V, t)$ is $O(\varepsilon^2)$-close to the mapping in time $2\varepsilon \sim |1/v(t)|$ along the trajectories of the autonomous system $v' = -(d_2^\bullet - d_1^\bullet)$, $t' = (d_2^\circ - d_1^\circ)/v$, where the prime stands for differentiation with respect to the new independent variable. This system reduces to the trivially integrable equation

$$d \ln v/dt = -(d_2^\bullet - d_1^\bullet)/(d_2^\circ - d_1^\circ)$$

which implies the required result.

Remark. If the functions f, f_1 and l in Examples 1, 4 and 5, or the functions d_i introduced in connection with the exact and simplified Ulam models, have a sufficiently high (but finite) degree of smoothness, all the results remain valid, but the exponentially small estimates should be replaced by power estimates. Theorem 4 is true on the assumption that the second derivatives of d_i° and d_i with respect to time are continuous; if only the first derivatives are continuous, the estimates $O(1/v(t))$ must be replaced by a quantity that converges uniformly in t to zero as $v(t) \to \infty$.

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